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NUMERICAL SOLUTION OF QUASILINEAR ATTRACTIVE TURNING POINT PROBLEMS

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Abstract — A quasilinear singularly perturbed boundary value problem with a turning point of the attractive type is considered. Analytic properties of its solution are established and used for the construction of a numerical method uniform in the perturbation parameter. The paper is an extension of a result obtained previously for the semilinear case.

1. INTRODUCTION

Singularly perturbed boundary value problems represent mathematical models for various important phenomena. Among them, the nonlinear turning point problems are of special interest. Some efficient numerical methods for such problems were given recently in [1] and [2], (see also the references therein). While [1] dealt with the quasilinear problem with a repulsive turning point, only a mild nonlinearity was considered in [2] for the attractive turning point case. This is also true for [3], where some assumptions from [2] have been relaxed. In this paper, we shall extend the results from [2] and [3] to the quasilinear case. We shall consider the problem:

$$-\varepsilon u'' - x\, b(x,u)\, u' + c(x,u) = 0, \quad x \in [-1,1],$$

with Dirichlet boundary conditions, small positive parameter ε and a positive function b. Other assumptions will be given later. They imply that the solution has an interior layer at $x = 0$. Because of that, the problem is close to the shock layer problems, such as Lagerstrom–Cole and Bürgers problems. However, it is simpler than general quasilinear shock problems since the position of the shock is known in advance. Still, by considering such a problem we make a step towards uniform numerical methods for the general shock problems. A different step towards the same goal was made in [4].

The technique we use in this paper is essentially the same as the one from [2] and [3]. That is why only the details which are different will be given. In Section 2 we shall analyse the continuous problem. We shall estimate how close the solution derivatives are to the corresponding derivatives of the solutions to the reduced problem (the problem with $\varepsilon = 0$). The numerical method will be presented in Section 3. It uses finite differences on a special discretization mesh which is dense in the layer. The discrete problem is uniformly stable with respect to ε in an L^1 discrete norm. Because of that, the error of the numerical solution will be estimated in that norm. However, numerical results in Section 4 will show uniform pointwise convergence as well.

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2. THE CONTINUOUS ANALYSIS

Consider the following quasilinear problem with a small positive perturbation parameter ε:

$$-\varepsilon u'' - x\, b(x,u)\, u' + c(x,u) = 0, \quad x \in I = [-1,1], \tag{1}$$

$$u(-1) = U_-, \quad u(1) = U_+. \tag{2}$$

We assume that the problem has a unique solution, which we denote by u_ε. Sufficient conditions for this assumption can be found in [5]. Moreover, let the following conditions hold throughout the paper:

(i) $c(x,u) = xc_1(x,u) + \varepsilon c_2(x,u)$,
(ii) $b(x,u),\ c_k(x,u), \in C^2(I \times \mathbf{R}), \qquad k = 1,2,$
(iii) $b(x,u) \geq b_* > 0, \qquad x \in I,\ u \in \mathbf{R},$
(iv) $|c_{k,u}(x,u)| \leq c^*, \qquad k = 1,2,\ x \in I,\ u \in \mathbf{R}.$

Moreover, we shall assume that ε is sufficiently small. The corresponding reduced problem has a discontinuous solution consisting of two smooth curves, u_+ and u_-, which satisfy:

$$b(x,u_\pm)\, u'_\pm - c_1(x,u_\pm) = 0, \quad u_\pm(\pm 1) = U_\pm.$$

In this section we shall estimate the quantities:

$$|(u_\varepsilon - u_\pm)^{(k)}(x)|, \qquad \text{for} \quad k = 0,1,2 \text{ and } x \in I_\pm,$$

where

$$I_- = [-1,0], \quad I_+ = [0,1].$$

By M we shall denote any (in the sense of $O(1)$) positive constant which is independent of ε. Some of these constants will be denoted by m, m_0, M_0, M_1, etc. Moreover, by μ we denote throughout $+\sqrt{\varepsilon}$. Finally, by $\|\cdot\|_\infty$, we denote the maximum norm in $C(I)$.

LEMMA 1. $|u_\varepsilon(x)| \leq M$.

PROOF. Let

$$p(x) = \begin{cases} |x|, & x \in I \backslash [-\mu,\mu]. \\ \frac{x^2}{2\mu} + \frac{\mu}{2}, & x \in [-\mu,\mu]. \end{cases}$$

We can easily verify that

$$p \in C^1(I), \ \max(|x|\,,\ \mu) \geq p(x) \geq \frac{1}{2}\max(|x|\,,\ \mu). \tag{3}$$

Next, consider the following Riccati initial value problem

$$P(\alpha) := \varepsilon\alpha' + xb_*\alpha + M_0\, p(x) + \varepsilon\,\alpha^2 = 0, \tag{4}$$

$$\alpha(0) = 0. \tag{5}$$

It has a uniformly bounded solution. Indeed, by applying Newton's method to (4) and (5) (cf. [1] or [6]), with the initial guess:

$$\alpha_0(x) = -\int_0^x \frac{M_0}{\varepsilon}\, p(t) \exp\left(-\frac{b_*}{2\varepsilon}(x^2 - t^2)\right) dt,$$

we can get that the conditions of the Newton–Kantorovich theorem are satisfied since:

$$\|\alpha_0(x)\|_\infty \leq M,$$
$$\|P'(\alpha_0)^{-1}\| \leq M\mu,$$
$$\|\alpha_1 - \alpha_0\|_\infty \leq M\mu,$$

Here, $||\cdot||$ is the operator norm corresponding to $||\cdot||_\infty$. Hence, there exists a solution $\alpha(x)$ to (4) and (5), such that

$$||\alpha - \alpha_0||_\infty \le M\mu,$$

thus α is bounded uniformly in ε. Furthermore, we have

$$x\,\alpha(x) \le 0 \qquad \text{for} \quad x \in I. \tag{6}$$

Indeed, because of maximum principle and

$$\varepsilon\alpha' + x\,b(x)\,\alpha = -M_0\,p(x) - \varepsilon\alpha^2 \le 0, \qquad \alpha(0) = 0,$$

we get that $\alpha(x) \le 0$ for $x \ge 0$ and $\alpha(x) \ge 0$ for $x \le 0$.

Let

$$\varphi(x) = \exp\left[\int_0^x \alpha(t)\,dt\right].$$

It holds that

$$0 < m \le \varphi(x) \le M, \qquad \varphi'(x) = \alpha(x) \quad \varphi(x) = O(1). \tag{7}$$

Note that φ is a solution to the following equation:

$$\varepsilon\varphi'' + xb_*\varphi' + M_0\,p(x)\,\varphi = 0.$$

Let us now consider an auxiliary problem:

$$\varepsilon u'' + xb(x, u_\varepsilon(x))u' - c(x,u) = 0, \tag{8}$$

$$u(-1) = U_-, \;\; u(1) = U_+, \tag{9}$$

and let us make the transformation $u(x) = z(x)\,\varphi(x)$. Then $z(x)$ satisfies :

$$\varepsilon z'' + [2\varepsilon\,\alpha(x) + xb(x, u_\varepsilon(x))]z' - \bar{c}(x,z) = 0, \tag{10}$$

$$z(-1) = \frac{U_-}{\varphi(-1)}, \;\; z(1) = \frac{U_+}{\varphi(1)}, \tag{11}$$

where

$$\begin{aligned} \bar{c}(x,z) &= -[\varepsilon\varphi''(x) + xb(x,u_\varepsilon(x))\,\varphi'(x)]\,\varphi(x)^{-1}z + \varphi(x)^{-1}c(x, z\varphi(x)) \\ &= M_0\,p(x)z - x[b(x,u_\varepsilon(x)) - b_*]\,\alpha(x)z + \varphi(x)^{-1}c(x, z\varphi(x)). \end{aligned}$$

Choosing M_0 sufficiently large and using (3) and (6) we have

$$\bar{c}_z(x,z) = M_0\,p(x) - x\alpha(x)\,(b(x,u_\varepsilon(x)) - b_*) + c_u(x, z\varphi(x)) \ge m_0 \max\,(|x|, \mu).$$

Hence, the problems (10) and (11) satisfies the maximum principle and therefore has a unique solution z, such that

$$|z(x)| \le |z(-1)| + |z(1)| + \max_{x\in I}\,\frac{|\varphi^{-1}c(x,0)|}{m_0 \max\,(|x|,\mu)} \le M.$$

This shows that (8) and (9) have a unique solution which is equal to u_ε. Then (7) completes the proof. ∎

Let

$$v_\pm(x) = u(x) - u_\pm(x), \qquad x \in I_\pm,$$

and

$$e(x) = \exp\left(-\frac{b_*}{4\varepsilon}\,x^2\right).$$

Then the following lemma holds:

LEMMA 2. $|v_\pm(x)| \le M[\mu + e(x)], \; x \in I_\pm$.

PROOF. We consider the case $x \in I_+$ only, since $x \in I_-$ is analogous. v_+ satisfies

$$\begin{aligned} \varepsilon v_+'' &+ x b(x, u_\varepsilon(x))\, v_+' + x[b(x, v_+ + u_+(x)) - b(x, u_+(x))]\, u_+'(x) \\ &- [c(x, v_+ + u_+(x)) - c(x, u_+(x))] = \varepsilon\, u_+''(x) - \varepsilon c_2(x, u_+(x)), \end{aligned} \tag{12}$$

$$v_+(0) = u_\varepsilon(0) - u_+(0), \quad v_+(1) = 0, \tag{13}$$

and it is a unique solution to this problem since u_ε and u_+ are unique solutions.

By means of the procedure similar to the proof of Lemma 1, making the transformation $v_+(x) = z_+(x)\,\varphi(x)$, where $\varphi(x)$ is defined as in Lemma 1, we have

$$L_+ z_+ := \varepsilon z_+'' + [x b(x, u_\varepsilon(x)) + 2\varepsilon\, \alpha(x)]\, z_+' - c^+(x, z_+) = \varphi(x)^{-1} O(\varepsilon),$$

$$z_+(0) = v_+(0), \quad z_+(1) = 0,$$

where

$$\begin{aligned} c^+(x, z_+) = \; & M_0\, p(x)\, z_+ - x[b(x, u_\varepsilon(x)) - b_*]\, \alpha(x)\, z_+ \\ & - \varphi(x)^{-1} x[b(x, u_+(x) + z_+\, \varphi(x)) - b(x, u_+(x))]\, u_+'(x) \\ & + \varphi^{-1}(x)\, [c(x, z_+\varphi(x) + u_+(x)) - c(x, u_+(x))]. \end{aligned}$$

It is easy to verify

$$c^+_{z_+}(x, z_+) = M_0\, p(x) - x\, \alpha(x)\, [b(x, u_\varepsilon(x)) - b_*] - x u_+'(x)\, b_u\, (x, u_+(x) + z_+\varphi(x))$$
$$+ c_u(x, z_+\varphi(x) + u_+(x)) \ge m_0 \max\, (|x|, \mu),$$

when M_0 is sufficiently large. Also, it holds that

$$x b(x, u_\varepsilon(x)) + 2\alpha(x)\, \varepsilon \ge \frac{b_*}{2}\, x,$$

for $x \ge 0$ and small ε, since $\alpha(x) = x\alpha'(\xi)$ and $\alpha'(\xi) = O(\mu^{-1})$. Then we construct the barrier function

$$w(x) = M_1\, \mu + M_2\, e(x)$$

to conclude that

$$\begin{aligned} L_+ w(x) &= M_2 \left[-\frac{b_*}{2}\, x + x\, b(x, u_\varepsilon(x)) + 2\varepsilon\, \alpha(x) \right] e'(x) \\ &- \frac{1}{2} b_* M_2\, e(x) - c^+(x, M_1\, \mu + M_2\, e(x)) \\ &\le -c^+(x, M_1\, \mu + M_2\, e(x)) \\ &= -c^+_{z_+}(x, \xi)(M_1\, \mu + M_2\, e(x)) \\ &\le -m_0 \max\, (|x|, \mu) M_1\, \mu \\ &\le -m_0\, M_1\, \varepsilon \le L_+\, z_+(x), \end{aligned}$$

when M_1 is sufficiently large. Similarly we have

$$L_+(-w(x)) \ge L_+\, z_+(x).$$

Noting that $w(0) \ge z_+(0) \ge -w(0)$, and that $w(1) = z_+(1) = 0$, by the maximum principle we have $|z_+(x)| \le w(x)$, hence

$$|v_+(x)| = |z_+|\, |\varphi| \le M\, w(x),$$

and the proof is completed. ∎

Next, we can prove:

Proof. Again, we shall give the proof for $v_+(x)$, $x \in I_+$. First, we rewrite (12) and (13) in the form:

$$\varepsilon v_+'' + F(x,v_+)' - F_x(x,v_+) - x b_u(x,\zeta)\, u_+{}' v_+ - c_u(x,\zeta)\, v_+ = O(\varepsilon), \tag{14}$$

with $F(x,v_+) = \int_0^{v_+} x\, b(x, s+u_+)\, ds$ and some real ζ. Integrating (14) from 0 to x_* (the point from $(0,\mu)$ given by $v_+'(x_*) = [v_+(\mu) - v_+(0)]/\mu$), we easily have

$$|v_+'(0)| \le \frac{M}{\mu}.$$

Next, from (12) and (13) we get

$$\begin{aligned} v_+'(x) \quad = \quad & \left\{ \varepsilon^{-1} \int_0^x \left[-t\, b_u(t,\zeta)\, u_+'(t)\, v_+(t) + c_u(t,\zeta)\, v_+(t) + O(\varepsilon) \right] \right. \\ & \left. \exp\left(\frac{B(t)}{\varepsilon}\right) dt + v_+'(0) \right\} \exp\left(-\frac{B(x)}{\varepsilon}\right), \end{aligned}$$

where $B(x) = \int_0^x t b(t, u_\varepsilon(t))\, dt$. Hence

$$|v_+'(x)| \le M(S_1 + S_2 + \mu^{-1} \exp\left(-\frac{B(x)}{\varepsilon}\right),$$

$$\begin{aligned} S_1 &= \int_0^x \left(1 + \frac{t}{\mu}\right) \exp\left[\frac{B(t) - B(x)}{\varepsilon}\right] dt, \\ S_2 &= \int_0^x \left(1 + \frac{t}{\varepsilon}\right) \exp\left(-\frac{b_*}{4\varepsilon} x^2\right) \exp\left[\frac{B(t) - B(x)}{\varepsilon}\right] dt. \end{aligned}$$

Using

$$\begin{aligned} \exp\left[\frac{B(t) - B(x)}{\varepsilon}\right] &= \exp\left[-\frac{1}{\varepsilon} \int_t^x s b(s, u_\varepsilon(s))\, ds\right] \\ &\le \exp\left[-\frac{b_*}{4\varepsilon}(x^2 - t^2)\right], \end{aligned}$$

we have

$$\begin{aligned} S_1 &\le M\mu, \\ S_2 &\le M \exp\left(-\frac{b_*}{4\varepsilon} x^2\right) \int_0^x \left(1 + \frac{t}{\varepsilon}\right) dt \\ &\le M\left[\mu + \frac{x^2}{\varepsilon} \exp\left(-\frac{b_*}{4\varepsilon} x^2\right)\right]. \end{aligned}$$

This completes the proof. ∎

We can proceed with the same technique (cf. [2], as well) to get the following estimates:

$$\begin{aligned} |v_\pm''(x)| &\le M\left[1 + \frac{|x|}{\varepsilon}\left(\frac{x^2}{\varepsilon} + \frac{1}{\mu}\right) e(x)\right], \qquad x \in I_\pm, \\ |u_\varepsilon'''(x)| &\le M\left[\frac{1}{\mu} + \frac{|x|}{\varepsilon}\left(\frac{|x|^3}{\varepsilon^2} + \frac{x^2}{\varepsilon} + \frac{|x|}{\mu^3} + \frac{1}{\mu}\right) e(x)\right]. \end{aligned}$$

Finally, in the same way as in [2], we can prove:

Theorem 1.

$$\begin{aligned} |(x v_\pm(x))'| &\le M(\mu V(x)), \qquad x \in I_\pm, \\ |(x v_\pm(x))''| &\le M(\mu + \mu^{-1} V(x)), \qquad x \in I_\pm, \\ \varepsilon |u_\varepsilon''(x)| &\le M(\varepsilon + V(x)), \qquad x \in I, \\ \varepsilon |u_\varepsilon'''(x)| &\le M(\mu + \mu^{-1} V(x)), \qquad x \in I \end{aligned}$$

where

$$V(x) = \exp\left(-\frac{a}{\mu}|x|\right),$$

with an arbitrary positive constant a independent of ε. These estimates will be necessary in the next section.

3. THE NUMERICAL METHOD

The numerical method is closely related to that from [2]. The same special non-equidistant mesh I^h is used. It has the following mesh points:

$$x_i = \lambda(t_i),\ \ t_i = -1 + \frac{2i}{n},\ \ i = 0(1)n, \qquad n = 2n_0,\ \ n_0 \in \mathbf{N},$$

where

$$\lambda(t) = \begin{cases} \omega(t) := \frac{\beta\mu t}{\gamma - t}, & t \in [0,\alpha] \\ \pi(t) := \delta(t-\alpha)^3 + \frac{1}{2}\omega''(\alpha)(t-\alpha)^2 + \omega'(\alpha)(t-\alpha) + \omega(\alpha), & t \in [\alpha, 1] \\ -\lambda(-t), & t \in [-1, 0] \end{cases}.$$

α is an arbitrary parameter from (0,1),

$$\gamma = \alpha + \mu^{\frac{1}{3}},$$

δ is determined from $\pi(1) = 1$, so that $\lambda \in C^2(I_\pm)$ and $\lambda \in C^1(I)$, and the parameter β is chosen from $(0, \gamma^{-1}(1-\alpha)^{-2}]$, (see [2]). Mesh generating functions such as λ have been used often, cf. [2,4,7] and the references therein. It may look as if λ is artificial, but its part ω is a suitable rational approximation to the logarithmic function representing the inverse of the interior layer function $V(x)$ for $x \geq 0$. Then π is just a smooth extension of ω.

Let

$$\begin{aligned} h_i &= x_i - x_{i-1}, \qquad i = 1(1)n, \\ \hbar_i &= \frac{1}{2}(h_i + h_{i+1}), \end{aligned}$$

and let w^h denote a mesh function on $I^h \setminus \{-1, 1\}$, which will be identified with the $\mathbf{R}^{n-1}$-vector:

$$w^h = [w_1, w_2, \ldots, w_{n-1}]^T,\ \ (w_i := w_i^h).$$

Moreover, let us introduce the following standard finite-difference operators:

$$\begin{aligned} D'_\pm w_i &= \frac{\pm(w_{i\pm1} - w_i)}{\hbar_i}, \\ D'' w_i &= \frac{[(w_{i-1} - w_i)/h_i + (w_{i+1} - w_i)/h_{i+1}]}{\hbar_i}. \end{aligned}$$

We shall use the following discrete L^1-norm:

$$\|w^h\|_1^h = \sum_{i=1}^{n-1} \hbar_i\, |w_i|.$$

For all this cf. [2,7]. Finally, in this section the constants M will be independent of I^h as well.

Before discretizing the problems (1) and (2), we shall rewrite (1) in the following conservation form:

where

$$\begin{aligned} f(x,u) &= \begin{cases} f_-(x,u), & x \in I_- \\ f_+(x,u), & x \in I_+ \end{cases}, \quad g(x,u) = \begin{cases} g_-(x,u), & x \in I_- \\ g_+(x,u), & x \in I_+ \end{cases}, \\ f_\pm(x,u) &= \int_{u_\pm(x)}^{u} xb(x,s)\,ds, \\ g_\pm(x,u) &= c(x,u) - xb(x,u_\pm(x))\,u'_\pm(x) + \int_{u_\pm(x)}^{u} (xb(x,s))_x\,ds \\ &= c(x,u) - xc_1(x,u_\pm(x)) + \int_{u_\pm(x)}^{u} (xb(x,s))_x\,ds. \end{aligned}$$

Then the discrete problem corresponding to (15) and (2) is given by:

$$T^h w_i = 0, \qquad i = 1(1)n-1, \tag{16}$$

where

$$\begin{aligned} T^h w_i &= \begin{cases} T^h_- w_i, & i = 1(1)n_0 \\ T^h_+ w_i, & i = n_0 + 1(1)n-1 \end{cases}, \\ T^h_\pm w_i &= -\varepsilon\, D'' w_i - D'_\pm\, f_\pm(x_i, w_i) + g_\pm(x_i, w_i), \end{aligned}$$

and where w_0 and w_n should be replaced by U_- and U_+, respectively. The discretization is a generalization of that for the mildly nonlinear case considered in [2] and [3].

Let us introduce the following assumption in addition to (i)–(iv):

$$(v)\;\; g_u(x,u) = (xb(x,u))_x + c_u(x,u) \ge g_* > 0, \qquad x \in I, \quad u \in \mathbf{R}.$$

(Note that in this case Theorem 9 from [5] guarantees that u_ε exists if additionally:

$$b^* \ge b(x,u), \qquad x \in I, \quad u \in \mathbf{R}\ ).$$

THEOREM 2. *The discrete probelem* (16) *has a unique solution* w^h_ε *and the following estimate holds:*

$$\|w^h_\varepsilon - u^h_\varepsilon\|^h_1 \le M\,\frac{1}{n}\,[\mu + \exp\,(-n)],$$

where

$$u^h_\varepsilon = [u_\varepsilon(x_1), u_\varepsilon(x_2), \ldots, u_\varepsilon(x_{n-1})]^T.$$

PROOF. The operator T^h is an M-operator and uniformly stable in $\|\cdot\|^h_1$. This can be proved by the same technique as the one from [2,7]. Then it remains to estimate the consistency error of T^h, but this can be done in the same way as in [2], by using Theorem 1 and properties of the mesh generating function λ. ∎

4. NUMERICAL RESULTS

We shall test our numerical method on the following example:

$$-\varepsilon u'' - x\,u\,u' + c(x) = 0, \qquad u(\pm 1) = U_\pm,$$

where $c(x)$ and $U_\pm$ are determined by taking the exact solution in the form

$$u_\varepsilon(x) = \exp\left(-\frac{x^2}{\varepsilon}\right) + x + 2.$$

The problem is chosen so that the solution behaves as described in Theorem 1. Note that c is not entirely of the form (i), and that (ii) and (v) hold only locally, since $u_\varepsilon(x) > 1$ for $x \in I$. Nevertheless, the numerical method works with

$$u_-(x) = u_+(x) = x + 2,$$

and the results show the same behaviour as those from [2] for a linear problem. The maximal pointwise error

$$E := \max_{1\le i\le n-1} |w_{\varepsilon,i} - u_\varepsilon(x_i)|$$

is uniform in ε with order 1, as well as the discrete L^1 error

$$E_1 := \|w_\varepsilon^h - u_\varepsilon^h\|_1^h.$$

Moreover, E_1 decreases together with ε, and the order is 1/2, which corresponds to Theorem 2. The results are given in the following table:

ε	n	50	100	200
1.–4	E	4.52–2	2.28–2	1.14–2
	E_1	1.02–3	5.23–4	2.66–4
1.–6	E	4.37–2	2.21–2	1.11–2
	E_1	1.04–4	5.39–5	2.74–5
1.–10	E	4.27–2	2.17–2	1.09–2
	E_1	1.08–6	5.57–7	2.83–7
1.–14	E	4.26–2	2.16–2	1.09–2
	E_1	1.09–8	5.63–9	2.89–9

There are 40% of the mesh steps within the interval $[-\mu, \mu]$. This is achieved in the same way as in [2]: $\alpha = 0.8$ is taken, and then the parameter β is chosen so that the desired percentage remains fixed for all values of ε and n.

The nonlinear system was solved by the Newton method, starting with the straight line through $(-1, U_-)$ and $(1, U_+)$. The method was stopped when the maximal difference between two successive iterations became less than 10^{-6}. Only 4 iterations were required for that.

REFERENCES

1. P. Lin, A numerical method for quasilinear singular perturbation problem with turning points, *Computing*, **46**, 155–164 (1991).
2. R. Vulanović, On numerical solution of a mildly nonlinear turning point problem, *RAIRO Math. Model. Numer. Anal.*, **24**, 765–784 (1990).
3. P. Lin, Remarks on a mildly nonlinear turning point problem, (to appear).
4. R. Vulanović, Continuous and numerical analysis of a boundary shock problem, *Bull. Austral. Math. Soc.*, **41**, 75–86 (1990).
5. J. Lorenz, Stability and monotonicity properties of stiff quasilinear boundary value problems, *Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.*, **12**, 151–175 (1982).
6. K. Niijima, On the behavior of solutions of a singularly perturbed boundary value problem with a turning point, *SIAM J. Math. Anal.*, **9**, 298–311 (1978).
7. R. Vulanović, Finite-difference schemes for quasilinear singuler perturbation problems, *J. Comput. Appl. Math.*, **26**, 345–365 (1989).